Orifice flow at high Knudsen numbers

By RODDAM NARASIMHA
Guggenheim Aeronautical Laboratory, California Institute of Technology, Pasadena, California

(Received 28 November 1960)

Several interesting features of the flow field in free-molecule flow through an orifice are discussed. An estimate is then made of the deviation of the mass flow $\dot{m}$ through the orifice from its limiting free-molecule value $\dot{m}^0$ for small departures from the limit. Using an iteration method proposed by Willis, it is shown that this deviation is of the first order in ϵ, the inverse Knudsen number, defined as the ratio of the radius of the hole to the mean free path in the gas at upstream infinity. An estimate of the coefficient is obtained making some reasonable assumptions about the three-dimensional nature of the flow, and the value so derived, giving $\dot{m} = \dot{m}^0(1 + 0{\cdot}25\epsilon)$, shows fair agreement with the measurements of Liepmann. It appears that 'nearly' free-molecular conditions prevail up to $\epsilon \sim 1{\cdot}0$.

1. Introduction

The flow through an orifice has been investigated recently by Liepmann (1961) in considerable detail, especially with a view to understanding the difficult problem of the transition from gasdynamic to gaskinetic flow. As pointed out by him, the finite geometry, the relative independence from surface interaction effects and the already available knowledge (however partial!) of the flow in the limiting cases of zero and infinite Reynolds number make the circular orifice a highly worthwhile configuration for study. Liepmann's measurements were made through a range of Knudsen numbers which effectively covered the 'compressible' transition from continuum to free-molecule flow, with the pressure ratios across the orifice infinity for all practical purposes.

The purpose here is to investigate theoretically the flow under nearly free-molecular conditions. In terms of an inverse Knudsen number ϵ, defined as the ratio of the radius R of the orifice to the mean free path λ_1 in the gas far upstream, this means that we are interested in flows at small ϵ. The free-molecule or gas-kinetic limit corresponds to $\epsilon \to 0$, and the mass flow rate $\dot{m}$ (per unit area) in this limit is well known from kinetic theory (Present 1958) to be

$$\dot{m}^0 = \tfrac{1}{4}\rho_1\bar{c}_1, \tag{1.1}$$

where ρ_1 is the density and $\bar{c}_1$ the mean molecular speed at upstream infinity. In the following, attention will primarily be confined to the mass flow rate $\dot{m}$ at non-zero ϵ, as this is the quantity on which measurements are available, though all other flow quantities can also be calculated from the results.

Before proceeding further it seems worthwhile to discuss some features of the free-molecule flow itself, and this is done in §2.

2. The free molecule limit

There is of course a definite flow field associated with even the free-molecule limit, and the field of any mean quantity, like density, pressure, etc., can be obtained by taking the appropriate moment of the molecular distribution function. Thus, if $f^0(\mathbf{v})$ denotes the distribution (the superscript 0 will always refer to free-molecule conditions), the corresponding density, 'gas' velocity, temperature and pressure, for instance, are given respectively by

$$\left.\begin{aligned} \rho^0 = mn^0 &= m\int f^0(\mathbf{v})\,D\mathbf{v}, \\ \mathbf{u}^0 &= \frac{1}{n^0}\int \mathbf{v}f^0(\mathbf{v})\,D\mathbf{v}, \\ T^0 = \frac{1}{2\mathscr{R}\beta^0} &= \frac{1}{3\mathscr{R}n^0}\int c^2 f^0(\mathbf{v})\,D\mathbf{v}, \\ p^0_{ij} &= -m\int c_i c_j f^0(\mathbf{v})\,D\mathbf{v}, \end{aligned}\right\} \tag{2.1}$$

where m is the mass of a molecule, $\mathbf{v}$ its velocity and $\mathbf{c} = \mathbf{v}-\mathbf{u}$ the 'peculiar' velocity; $\mathscr{R}$ is the gas constant and $D\mathbf{v}$ is an element of volume in $\mathbf{v}$-space.

The function f^0 is obtained from the consideration that in the free-molecule limit there are no inter-molecular collisions. As the pressure ratio across the orifice p_1/p_2 tends to infinity, there will thus be no molecules travelling upstream across the orifice at all, and the distribution function at a point like $P(\mathbf{x})$ (see figure 1) will be Maxwellian everywhere in velocity space except in the 'vacant cone' C, which is the backward cone subtended by the orifice at P. The Maxwellian part corresponds of course to conditions at upstream infinity; thus (with subscript 1 denoting these conditions) we have

$$\begin{aligned} f^0(\mathbf{v}) &= 0 \quad \text{in} \quad C \\ &= n_1\left(\frac{\beta_1}{\pi}\right)^{\frac{3}{2}} e^{-\beta_1 v^2} \quad \text{everywhere else.} \end{aligned} \tag{2.2}$$

The function f^0 is similar downstream of the orifice too, except that now C is the major part of velocity space, and the backward cone is a 'full' Maxwellian. All across the orifice the cone becomes a half-space, and hence f^0 and all the mean quantities are constant.

The number density is relatively easy to calculate, being given (in non-dimensional form) by

$$N(\mathbf{x}) = \frac{n^0(\mathbf{x})}{n_1} = \int_0^\infty \left(\frac{\beta_1}{\pi}\right)^{\frac{3}{2}} e^{-\beta_1 v^2}(4\pi-\Omega)\,v^2\,dv = \frac{1-\Omega(\mathbf{x})}{4\pi}, \tag{2.3}$$

where $\Omega(\mathbf{x})$, the solid angle subtended by the orifice at $\mathbf{x}$, is

$$\Omega = \int_0^{2\pi} z\,d\phi \int_0^R \frac{R'\,dR'}{(\eta^2+z^2-2\eta\cos\phi R'+R'^2)^{\frac{3}{2}}}, \tag{2.4}$$

writing η for the cylindrical radius at $\mathbf{x}$.

The evaluation of this integral is fairly straightforward but somewhat tedious, and leads to the expression

$$\frac{\Omega}{4\pi} = \tfrac{1}{4}\{\Lambda_0(\alpha_1, \alpha_2) - \cos\alpha_1 \sin\alpha_2 F_0(\alpha_1)\}, \tag{2.5}$$

where Λ_0 is Heuman's lambda function and $\frac{1}{2}\pi F_0(\alpha_1)$ is the complete elliptic integral of the first kind, $K(\sin\alpha_1)$. Heuman (1941) has discussed the properties of

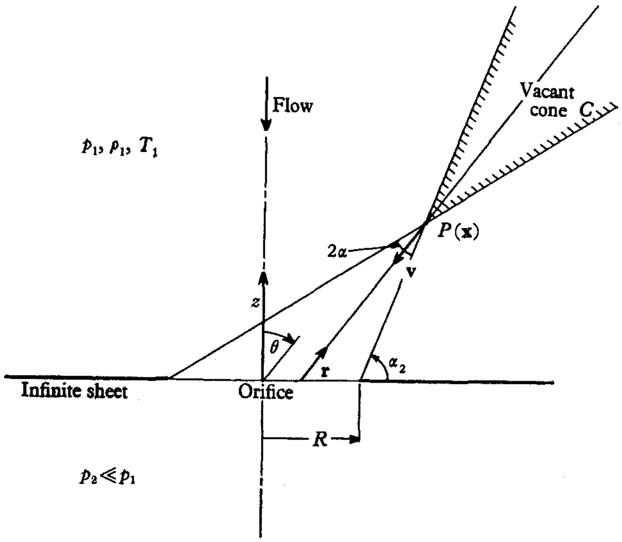

FIGURE 1. Diagram explaining notation.

the function Λ_0, and has also tabulated both Λ_0 and F_0. The arguments α_1, α_2 are given, in terms of polar co-ordinates (r, θ) from the centre of the orifice, by

$$\sin\alpha_1 = \left(\frac{4r\sin\theta}{r^2 + 2r\sin\theta + 1}\right)^{\frac{1}{2}}, \quad \tan\alpha_2 = \frac{r\cos\theta}{r\sin\theta - 1}.$$

(Distances have all been non-dimensionalized by dividing by R.) A relation corresponding to (2.5) has been derived by Sadowsky & Sternberg (1950) for the stream function of a source ring.

The density field calculated from (2.4) and (2.5) has been plotted in figure 2 for the region upstream of the orifice. The field downstream is obtained easily by making use of the symmetry

$$N(r, \pi - \theta) = 1 - N(r, \theta), \tag{2.6}$$

which follows from the nature of f^0 discussed before. In the plane of the orifice $\Omega = 2\pi$, so $N = \frac{1}{2}$. On the axis $\alpha_1 = 0$ and $\tan\alpha_2 = -r$, and (2.5) reduces to

$$\frac{\Omega}{4\pi} = \tfrac{1}{2}(1 - \sin\alpha_2);$$

hence

$$N = N(z) = \frac{\frac{1}{2}(z + \sqrt{(1+z^2)})}{\sqrt{(1+z^2)}} = \cos^2\tfrac{1}{2}\alpha, \tag{2.7}$$

if 2α is the included angle at the vertex of C.

The gas velocity can similarly be expressed as

$$\mathbf{U}(\mathbf{x}) = \sqrt{\beta_1}\,\mathbf{u}(\mathbf{x}) = -\frac{\sqrt{\beta_1}}{n^0(\mathbf{x})}\int_C n_1\left(\frac{\beta_1}{\pi}\right)^{\frac{3}{2}} e^{-\beta_1 v^2}\,\mathbf{v}\,D\mathbf{v}. \tag{2.8}$$

This integral unfortunately turns out to be too complicated to express analytically, and we must at present content ourselves with calculating it along $\theta = 0$ and 90^0. Along the axis we can show quite easily that

$$U(z) = -U_z(z) = \frac{1}{\sqrt{\pi}}\left(1-\frac{z}{\sqrt{(1+z^2)}}\right) = \frac{2}{\sqrt{\pi}}\sin^2\tfrac{1}{2}\alpha; \tag{2.9}$$

across the orifice U is constant and equal to $-U_z(0) = 1/\sqrt{\pi}$ (or $u^0 = -u_z^0(0) = \frac{1}{2}\bar{c}_1$). However, it is not difficult to guess the qualititative behaviour of U for other

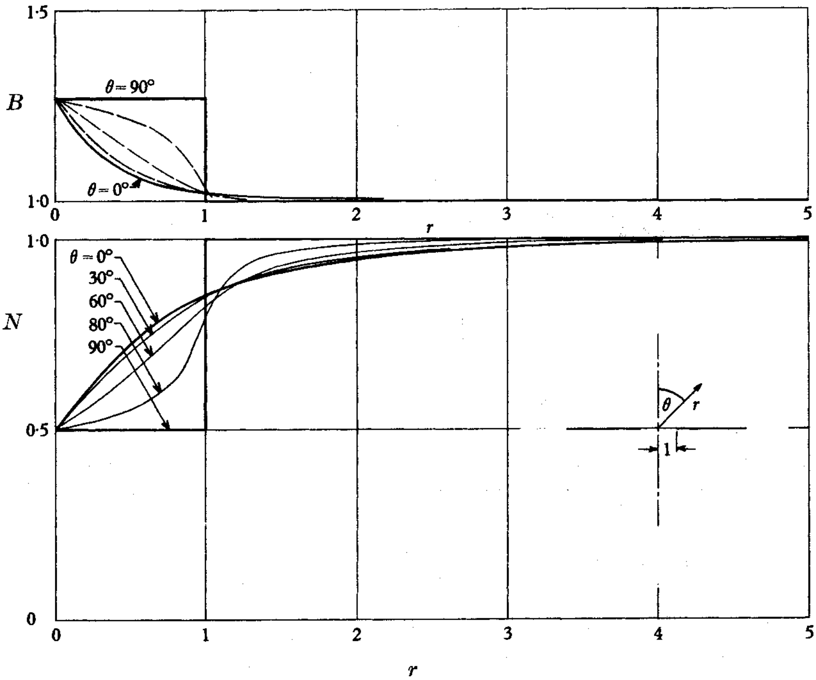

FIGURE 2. N is antisymmetric about $r = 0$, $N = \frac{1}{2}$. Full lines show exact results and dashed lines represent rough estimates.

values of θ, using N as a guide. It is interesting to note that we can immediately obtain the free-molecule mass flow from the mean field as

$$m^0 = \rho^0(-u_z^0) = (\tfrac{1}{2}\rho_1)(\tfrac{1}{2}\bar{c}_1) = \tfrac{1}{4}\rho_1\bar{c}_1.$$

A little thought shows that U, like N, is antisymmetric about the orifice and has the value

$$U(r, \pi-\theta) = 2\pi^{-\frac{1}{2}} - U(r,\theta). \tag{2.10}$$

Thus, far downstream the gas velocity is finite and tends to $\bar{c}_1$, but the density of course drops off to zero. It should be noticed that though the velocity field is antisymmetric about the plane $z = 0$ the stream line pattern will be perfectly symmetric as it depends only on the geometry of the cone C.

The temperature field is obtained from the relation $\mathscr{R}T^0(\mathbf{x}) = \frac{1}{3}\overline{c^2} = \frac{1}{3}(\overline{v^2} - u^2)$. But

$$\overline{v^2}(\mathbf{x}) = \frac{1}{n^0(\mathbf{x})}\int_0^\infty n_1\left(\frac{\beta_1}{\pi}\right)^{\frac{3}{2}} e^{-\beta_1 v^2} v^2(4\pi - \Omega)\, v^2 dv = (\overline{v^2})_1,$$

hence

$$\mathscr{R}T^0(\mathbf{x}) = \mathscr{R}T_1 - \tfrac{1}{3}u^2(\mathbf{x}),$$

or

$$\frac{T^0(\mathbf{x})}{T_1} = \frac{\beta_1}{\beta^0(\mathbf{x})} = \frac{1}{B} = 1 - \tfrac{2}{3}U^2(\mathbf{x}). \tag{2.11}$$

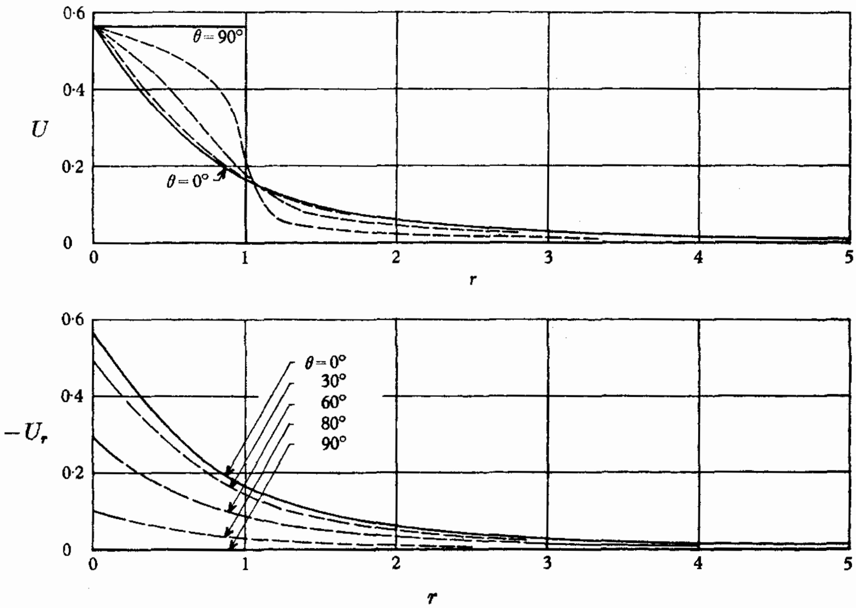

FIGURE 3. U is antisymmetric about $r = 0$, $U = \pi^{-\frac{1}{2}}$. Full lines show exact results and dashed lines represent rough estimates

Thus the temperature is very simply related to the velocity. It drops slightly from T_1 to $\{1 - 2/(3\pi)\}\,T_1 = 0{\cdot}7879T_1$ at the orifice and then sharply decreases to $T_2 = \{1 - 8/(3\pi)\}\,T_1 = 0{\cdot}1512T_1$ far downstream. Along the axis, we get from (2.9)

$$B = B(z) = \left[1 - \tfrac{2}{3}\pi^{-1}\frac{1}{(z^2+1)\{z+\sqrt{(z^2+1)}\}^2}\right]^{-1} = (1 - \tfrac{8}{3}\pi^{-1}\sin^4\tfrac{1}{2}\alpha)^{-1}. \tag{2.12}$$

From (2.1) the pressure tensor can be written as

$$p_{ij}^0 = -\rho^0(\overline{v_i v_j} - u_i u_j).$$

The thermodynamic pressure $p = \frac{1}{3}p_{ii}$ can of course be obtained from the equation of state $p = \rho\mathscr{R}T$; some of the other components can also be easily evaluated but the results will not be quoted here.

The quantities N, B, U and U_r (the radial component of U which will turn out to be of interest later) are plotted in figures 2 and 3. As an illustration of the kind of flow pattern to be expected in free-molecule flow the streamlines have been worked out for a slit and are shown in figure 4. The calculations here are exactly the same as for a circular orifice but have the advantage that they lead to closed expressions for all quantities; in particular

$$N = N(r,\theta) = 1 - \alpha/\pi,$$
$$U = U(r,\theta) = \sin\alpha/2\sqrt{\pi}(1-\alpha/\sqrt{\pi}),$$

where $2\alpha = \tan^{-1}\{2r\cos\theta/(r^2-1)\}$ is again the included angle at P. The temperature is still given by (2.11). It can be easily shown that the vector $\mathbf{U}$ bisects the angle 2α at P, and this suggests an obvious geometric construction for drawing

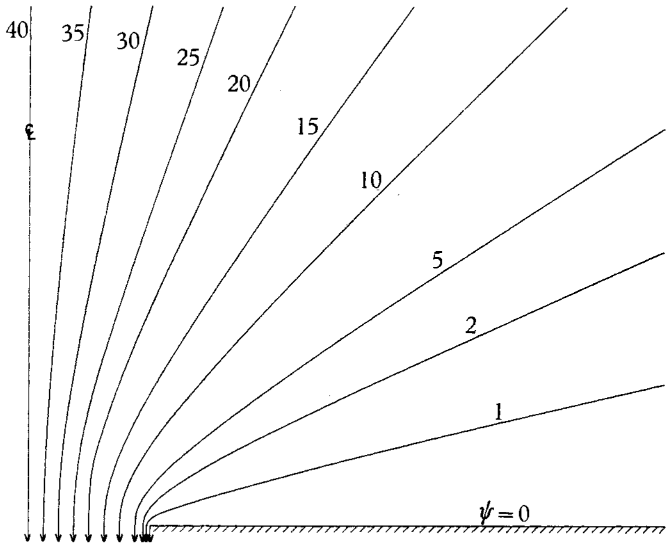

FIGURE 4. Streamlines in free-molecule flow through a slit. The pattern is symmetric about the plane of the slit and the centre-line.

the streamlines. As remarked earlier the streamlines are symmetric about the plane of the orifice. The effect of the wall in slowing down the flow is quite impressive.

One might make a few remarks here on the general validity of the above picture when the flow takes place from one *finite* reservoir into another. Conditions upstream are obviously not going to be affected too much if the linear size of the reservoir is sufficiently large compared to the orifice diameter and the mean free path. Downstream of the orifice, however, the presence of a wall tends to bring the gas temperature back to T_1, while we saw above that the temperature T_2 in the case of an infinite reservoir is much less than T_1. Thus heat transfer at the walls becomes important, and the mean field will presumably be rather different from the one discussed here. The point has been discussed by Liepmann (1961), and we will return to it briefly later.

3. The nearly free-molecule flow

For small departures from the free-molecule limit we can obtain the true distribution function f by perturbing f^0. This function f is governed by Boltzmann's equation, which for the steady flow of a monatomic gas (the only one we consider here) in the absence of any external forces is (Chapman & Cowling 1952)

$$\mathbf{v}.\frac{\partial f(\mathbf{v},\mathbf{x})}{\partial \mathbf{x}} = \int f(\mathbf{v}')f(\mathbf{w}')\,gI\,d\Omega D\mathbf{w} - f(\mathbf{v})\int f(\mathbf{w})\,gI\,d\Omega D\mathbf{w}$$
$$= \mathscr{G}(f) - f\mathscr{L}(f). \tag{3.1}$$

Here g denotes the relative velocity between two molecules which have initial velocities $\mathbf{v}$, $\mathbf{w}$ and final velocities $\mathbf{v}'$, $\mathbf{w}'$, and I the differential cross-section for scattering into the solid angle $d\Omega$. For convenience we denote the integral operators by $\mathscr{G}$ and $\mathscr{L}$, the 'gain' and 'loss' operators on f.

To tackle the Boltzmann equation directly seems a rather hopeless task; we obtain here an approximate solution f^1 for small ϵ by an iteration procedure suggested by Willis (1958) in which we replace $\mathscr{G}(f)$ and $\mathscr{L}(f)$ by $\mathscr{G}(f^0)$ and $\mathscr{L}(f^0)$ and write

$$\mathbf{v}.\frac{\partial f^1(\mathbf{v},\mathbf{x})}{\partial \mathbf{x}} = \mathscr{G}(f^0) - f^1\mathscr{L}(f^0),$$

or, considering flow upstream of the orifice as shown in figure 1,

$$-v\frac{\partial f^1(\mathbf{v},\mathbf{x})}{dr} + f^1\mathscr{L}(f^0) = \mathscr{G}(f^0). \tag{3.2}$$

This is an ordinary differential equation with $\mathbf{v} = -v\hat{\mathbf{r}}$ as a parameter, for each value of which we require a solution of (3.2). (Note that r, at this stage, is not measured from the centre.) The solution of the equation, using the boundary condition that $f = f_1$ at $r = \infty$, is

$$f^1(\mathbf{v},\mathbf{x}) = f_1 \exp\left\{-\int_\infty^r \frac{\mathscr{L}(f^0)}{-v}dr'\right\} + \int_\infty^r \frac{\mathscr{G}(f^0)}{-v}\exp\left\{-\int_{r'}^r \frac{\mathscr{L}(f^0)}{-v}dr''\right\}dr'. \tag{3.3}$$

As $\mathscr{L}(f)/v$ is like $1/\lambda$, the first term obviously vanishes for any finite r, leaving us with

$$f^1(\mathbf{v}, r=0) = \int_0^\infty \frac{\mathscr{G}(f^0)}{v}\exp\left\{-\int_0^{r'}\frac{\mathscr{L}(f^0)}{v}dr''\right\}dr'. \tag{3.4}$$

It is hoped, of course, that this first iterate gives a good approximation to the true perturbation. Incidentally (3.4) has an immediate and obvious physical interpretation, and could almost have been written down straightaway; for $\mathscr{G}(f^0)/v$ is the number of molecules which, after collision in unit distance, are travelling with velocity v, and the exponential multiplying it is the probability of such molecules reaching the orifice ($r = 0$).

The simplest molecular model we can use is the one due to Krook (Bhatnagar, Gross & Krook 1954, Krook 1959), namely,

$$\left.\begin{aligned} \mathscr{L}(f^0) &= An^0, \\ \mathscr{G}(f^0) &= A(n^0)^2(\beta^0/\pi)^{\frac{3}{2}}\exp\{-\beta^0(\mathbf{v}-\mathbf{u}^0)^2\}, \end{aligned}\right\} \tag{3.5}$$

where n^0, $\mathbf{u}^0$ and β^0 are all given by (2.1). We choose A here to be a constant equal to $\bar{c}_1/n_1\lambda_1$ so that the number of collisions (per unit volume and unit time) when the distribution is completely Maxwellian, namely $\frac{1}{2}(n_1\bar{c}_1/\lambda_1)$, agrees with the number given by Krook's model, which is $\frac{1}{2}A(n^0)^2$. Making these substitutions in (3.4) and introducing

$$\mathbf{v}\sqrt{\beta_1} = \mathbf{V}, \quad \epsilon/\sqrt{\pi} = \epsilon'$$

and other non-dimensional quantities already used in §2, we get

$$f^1(\mathbf{V}, r=0) = f_1 2\epsilon' \int_0^\infty N^2 B^{\frac{3}{2}} \exp\{\mathbf{V}^2 - B(\mathbf{V}-\mathbf{U})^2\} \exp\left\{-\frac{\epsilon'}{V}\int_0^r 2N(r)\,dr'\right\}\frac{dr}{V}. \tag{3.6}$$

In principle this integral can now be evaluated for each velocity vector $\mathbf{V}$ at each point in the plane of the orifice, for all the quantities in the integrand refer to free-molecule flow and can be calculated as in §2.

However the explicit and analytical calculation of the integral is rather difficult. But we can still get a good estimate of the result by making a few simple approximations. First we assume that the flow is uniform across the orifice: this is strictly true in free-molecule flow, and should be a good approximation in nearly free-molecule flow too. This means that we evaluate the integral at the centre of the orifice, and assume it is the same all across it.

We also notice, from figures 2 and 3 and from the discussion in §1, that the values N, B and U—but not of U_r, which appears in $(\mathbf{V}-\mathbf{U})^2$ in (3.6)—at $r=0$ and ∞ are independent of θ, though their variation in between is a function of θ. It is obvious, therefore, that the integral in (3·6) is similar for all rays, though its actual numerical value will vary somewhat. We shall in fact make use of this feature in what follows.

It is convenient for purposes of analysis to rewrite (3.6) as

$$f^1(\mathbf{V}, r=0) = f_1 2\epsilon' \int_0^\infty g_1(r)\, g_2(r, V)\, H(r, V; \epsilon')\frac{dr}{V}, \tag{3.7}$$

where

$$\left.\begin{aligned} g_1(r) &= N^2 B^{\frac{3}{2}} e^{-BU^2}, \\ g_2(r, V) &= \exp\{-2BU_r V + (1-B)V^2\} \\ &= \exp\{h_1(r)V + h_2(r)V^2\}, \\ H(r, V; \epsilon') &= \exp\left\{-\frac{\epsilon'}{V}\int_0^r 2N(r')\,dr'\right\} = \exp\left\{-\frac{\epsilon'}{V}h(r)\right\}. \end{aligned}\right\} \tag{3.8}$$

It is particularly instructive to work out the integral in (3.7) along the axis, $\theta = 0$, where N, B, U and $U_r\,(=-U)$ are all analytically known from (2.7), (2.9) and (2.12); also

$$h(z) = \int_0^z 2N(z)\,dz = z + \sqrt{(z^2+1)} - 1. \tag{3.9}$$

We have therefore

$$f^1(-V\hat{\mathbf{z}}, z=0) = f_1 2\epsilon' \int_0^\infty g_1(z)\, g_2(z, V)\, H(z, V, \epsilon')\frac{dz}{V}. \tag{3.10}$$

We want the lowest order term in ϵ' from this expression. Now it is shown in Appendix II that the integral for the mass flow (or, for that matter, any moment of f^1) can be split in a way which corresponds exactly to splitting the distribution function in (3.10) as follows:

$$f^1(-V\hat{\mathbf{z}}, z=0) = f_1 2\epsilon'\left[\int_0^\infty (g_1 g_2 - 1)\frac{dz}{V} + \int_0^\infty H\frac{dz}{V}\right]. \tag{3.11}$$

The error committed in this splitting is $o(\epsilon')$; the lowest order term in ϵ' turns out to be $O(\epsilon')$ and (3.11) reduces, correct to $O(\epsilon')$, to

$$f^1(-V\hat{\mathbf{z}}, z=0) = f_1\left\{1+\frac{2\epsilon'}{V}\left[\tfrac{1}{2}+\int_0^\infty (g_1 g_2 - 1)\,dz\right]\right\} \tag{3.11a}$$

using the integral evaluated in Appendix I. We expand g_2 as a series in V, obtaining

$$g_2 = 1 + Vh_1 + V^2(h_1^2/2!+h_2) + V^3(h_1^3/3!+h_1h_2) + V^4(h_1^4/4!+h_1^2h_2/2+h_2^2/2)+\ldots.$$

Thus we can write
$$f^1(-V\hat{\mathbf{z}}) = f_1\left\{1+\frac{2\epsilon'}{V}\phi_0(V)\right\}, \tag{3.12}$$

where
$$\phi_0(V) = \sum_{n=0}^{\infty} A_n V^n,$$

$$A_0 = \tfrac{1}{2}+\int_0^\infty (g_1-1)\,dz,\quad A_1 = \int_0^\infty g_1h_1\,dz,\quad A_2 = \int_0^\infty (g_1h_1^2/2!+g_1h_2)\,dz,\ldots. \tag{3.13}$$

From arguments given previously we expect the distribution function to look similar on other rays, so we may assume

$$f^1(\mathbf{V}, r=0) = f_1\left\{1+\frac{2\epsilon'}{V}\phi(\mathbf{V})\right\}. \tag{3.14}$$

Looking at (3.6), (3.7) and (3.8) again, and surmising the variation of the integrand with θ from figures 2 and 3, we see (as remarked earlier) that N, B and U vary between the same limits at $r=0$ and $r=\infty$ for all θ, and so the effect on the integral of their being functions of θ must be small. $\left(\text{For instance } \int_0^\infty \{1-N(r)\}\,dr \text{ is exactly the same for } \theta=0 \text{ and } \theta=90^\circ.\right)$ The predominant effect of the dependence on θ of the integral appears through U_r, which varies quite widely with θ. Near the origin we obviously have $U_r = U_z(z=r)\cos\theta$ as $|U_z| \simeq U$. For $r \gg 1$ it can be easily shown from (2.8) that

$$-U_r \simeq U \simeq \frac{2}{\sqrt{\pi}}\frac{\Omega}{4\pi};$$

but $\Omega \simeq \pi\cos\theta/r^2$, so again $U_r \simeq U_z(z=r)\cos\theta$. Thus, it is a very good approximation to take
$$U_r(r,\theta) \simeq U_z(z=r)\cos\theta. \tag{3.15}$$

Next we notice that h_2 accounts for a relatively small contribution to the final answer in comparison with h_1 (h_2 varies from about $-0{\cdot}27$ to zero as r goes from 0 to ∞, and appears only in V^2 and higher order terms); hence it appears that, approximately, the coefficients of V^n in (3.13) are multiplied by corresponding powers of $\cos\theta$ on other rays. We can therefore take

$$\phi(\mathbf{V}) \simeq \phi_0(V\cos\theta), \tag{3.16}$$

which also amounts to putting $g_2(r, V) = g_2(z, V\cos\theta)$. Using this in (3.14) we get the final result

$$f^1(\mathbf{V}, r=0) = f_1\left\{1+\frac{2\epsilon'}{V}\sum_{n=0}^{\infty} A_n V^n \cos^n\theta\right\}. \tag{3.17}$$

Results

The coefficients A_n have been calculated numerically for $n = 0$ to 5, and have the following values:

$$\left.\begin{aligned} A_0 &= -0{\cdot}3944, \quad A_1 = +0{\cdot}7549, \quad A_2 = +0{\cdot}1055, \\ A_3 &= +0{\cdot}0502, \quad A_4 = -0{\cdot}0045, \quad A_5 = -0{\cdot}0013. \end{aligned}\right\} \tag{3.18}$$

The mass flow through the orifice is easily obtained as

$$\dot{m} = m \int (-v_z) f(\mathbf{v})\, D\mathbf{v}. \tag{3.19}$$

Substituting from (3.17) we get

$$\begin{aligned} \dot{m} &= (m/\beta_1^2) \int_0^\infty \int_0^{\frac{1}{2}\pi} f_1\{V\cos\theta + 2\epsilon' \cos\theta\, \Sigma A_n V^n \cos^n\theta\}\, 2\pi V^2 \sin\theta\, d\theta\, dV \\ &= \tfrac{1}{4}\rho_1 \bar{c}_1 \left[1 + 4\epsilon' \Sigma \frac{A_n}{n+2} \left(\frac{n+1}{2}\right)!\right]. \end{aligned} \tag{3.20}$$

Using the values of A_n given by (3.18) we get†

$$\dot{m} = \tfrac{1}{4}\rho_1 \bar{c}_1 (1 + 0{\cdot}25\epsilon). \tag{3.21}$$

The coefficient 0·25 obtained here is somewhat less than the value 0·26 given in a preliminary report (Narasimha 1960) where the calculations were based on an assumption slightly different from the one used here in (3.16), namely

$$\Phi(\mathbf{V}) \equiv \phi(\mathbf{V})/V = \Phi_0(V\cos\theta), \tag{3.22}$$

and h_2 was ignored. The approximation (3.16) is somewhat more consistent, but apparently the result obtained is not significantly different. No claim of great accuracy is made for the coefficient; nor indeed would it be justified, considering the approximate nature of the calculations.

Correction for backflow

The backflow through the orifice is nil under free molecule conditions, but one may in general expect some backflow when ϵ is finite, even though the pressure ratio p_1/p_2 is still infinite. Using exactly the same procedure as before, one can write down the distribution function for molecules travelling upstream as

$$f^1(V\hat{\mathbf{z}}, z=0) = f_1 2\epsilon' \int_0^{-\infty} g_1(z)\, g_2(z, V)\, H(z, V; \epsilon') \frac{dz}{V}$$

where g_1, g_2 and H are still given by (3.8), but of course one now has to use the downstream field of N, B and U, obtained as shown in § 2. Both h_1 and h_2 are then negative, and $h(z) \to 1$ as $z \to -\infty$. By repeating the analysis made for the upstream integral one concludes that the highest order term in $f^1(V\hat{\mathbf{z}}, z=0)$ is $O(\epsilon')$, and given by

$$f^1(V\hat{\mathbf{z}}, z=0) = f_1 2\epsilon' \int_0^{-\infty} g_1(z)\, g_2(z, V) \frac{dz}{V}.$$

† I have to thank Dr Willis for pointing out a numerical mistake at this point in an earlier draft.

The approximation (3.16) for the radial component U_r is now rather crude, but conservative in the sense that it overestimates the backflow. The correction to the coefficient (3.21) can again be pressed as

$$a' = 4\Sigma \frac{A'_n}{n+2}\left(\frac{n+1}{2}\right)!$$

where

$$A'_0 = \int_0^{-\infty} g_1 dz, \quad A'_1 = \int_0^{-\infty} g_1 h_1 dz, \quad A'_2 = \int_0^{-\infty}\left(\frac{g_1 h_1^2}{2!}+g_1 h_2\right) dz, \ldots.$$

The coefficients alternate in sign, and using the Euler–Maclaurin technique one arrives at an upper bound of $-0{\cdot}03$ for a', i.e. the coefficient in (3.21) would be reduced at most to 0·22 by backflow.

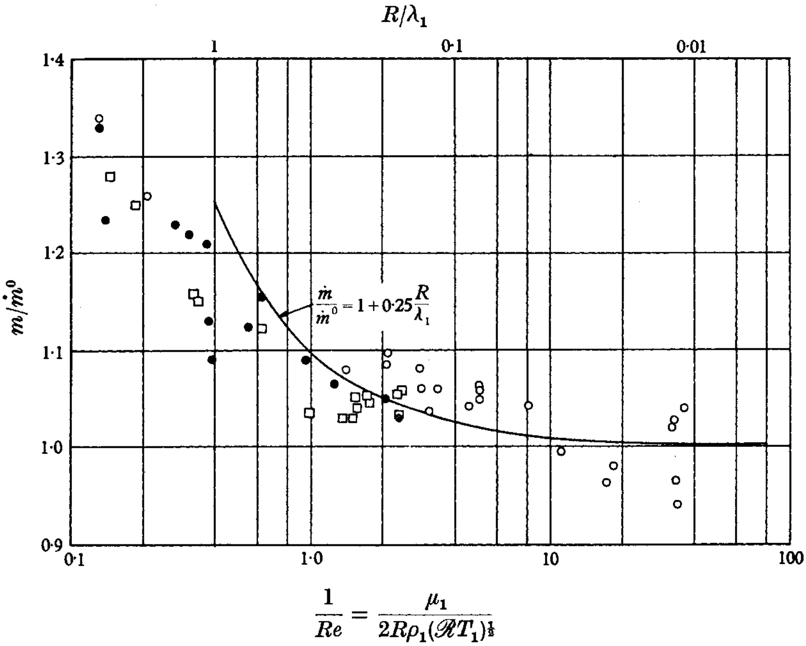

FIGURE 5. Comparison of theory with experimental data. Liepmann's measurements: ●, argon; ○, helium; □, nitrogen. —, values calculated with $\mu_1 = \frac{1}{2}\rho_1\bar{c}_1\lambda_1$.

Comparison with experiment

Figure 5 shows the experimental data of Liepmann, and also the result (3.21), which has been plotted assuming that the viscosity $\mu_1 = \frac{1}{2}\rho_1\bar{c}_1\lambda_1$. The backflow correction has not been included first because it is small and would not make much difference; secondly, because its exact value is rather uncertain in the case of flow into a finite reservoir, where the flow field is affected somewhat by the presence of walls as discussed in §2.

Figure 5 shows reasonable agreement between theory and experiment (notice that a small backflow correction would make the agreement even better), and it seems safe to conclude that: (i) the prediction of the theory that the departure from free-molecule flow is linear in ϵ with a coefficient of the order of 0·25 is borne out by the experiments; and (ii) 'nearly' free-molecule conditions prevail up to $\epsilon \sim 1{\cdot}0$.

This work has been supported by the Office of Naval Research, contract no. N-onr 220-21 Task 21. I am grateful to Dr H. W. Liepmann and Dr J. D. Cole for discussing the problem with me and for their enthusiastic encouragement. I also want to thank Mrs Dorothy Diamond for making most of the computations.

Appendix I

We evaluate here an integral which we will need, namely:

$$\chi(a) = a\int_0^\infty \exp\{-a[z+\sqrt{(z^2+1)}]\}\,dz = \int_0^\infty \exp\{-[\zeta+\sqrt{(\zeta^2+a^2)}]\}\,d\zeta.$$

This transforms, using $\xi = \zeta+\sqrt{(a^2+\zeta^2)}$, $d\zeta/d\xi = (\xi^2+a^2)/2\xi^2$, to

$$\chi(a) = \int_a^\infty \frac{\xi^2+a^2}{2\xi^2}\,e^{-\xi}\,d\xi.$$

Integrating successively by parts, we can reduce this to

$$\chi(a) = \tfrac{1}{2}\{(1+a)\,e^{-a}-a^2\mathrm{Ei}(a)\},$$

where $\mathrm{Ei}(a)$ is the exponential integral, with the following expansion for small a (Bromwich, 1926):

$$\mathrm{Ei}(a) = -0{\cdot}5772\ldots-\ln a+a+\ldots.$$

Thus χ for small a is

$$\chi(a) = \tfrac{1}{2}+O(a^2\ln a) = \chi(0)+o(a). \tag{A-1}$$

Appendix II

From the discussion which led to (3.16) it was concluded that

$$g_2(r, V) \simeq g_2(z, V\cos\theta),$$

and that a good approximation to the distribution function is

$$f^1(\mathbf{V}, r=0) = f_1\,2\epsilon'\int_0^\infty g_1(z)\,g_2(z, V\cos\theta)\,H(z, V;\epsilon')\frac{dz}{V}. \tag{A-2}$$

Hence the mass flow, expressed as an integral, is (from 3.19)

$$\begin{aligned}\dot{m} &= 2\epsilon'\left(\frac{\rho_1}{\beta_1^2}\right)\left(\frac{\beta_1}{\pi}\right)^{\frac{3}{2}}\iiint e^{-V^2}g_1(z)\,g_2(z, V\cos\theta)\,H(z, V;\epsilon')\,2\pi V^2\cos\theta\sin\theta\,d\theta\,dz\,dV\\ &= 2\rho_1\bar{c}_1\epsilon'\int_0^\infty\int_0^\infty\int_0^{\frac{1}{2}\pi} V^2e^{-V^2}g_1(z)\,g_2(z, V\cos\theta)\,H(z, V;\epsilon')\cos\theta\sin\theta\,d\theta\,dz\,dV.\end{aligned} \tag{A-3}$$

Consider first the integrations with respect to z and V. Putting

$$g_1(z)\,g_2(z, V\cos\theta) = G(z, V\cos\theta),$$

we write

$$J(\epsilon') = \epsilon'\iint V^2e^{-V^2}(G-1)\,H\,dz\,dV+\epsilon'\iint V^2e^{-V^2}H\,dz\,dV = J_1(\epsilon')+J_2(\epsilon'), \tag{A-4}$$

the reason for the splitting being that $(G-1)$ is integrable in z. We want to extract from $J(\epsilon')$ the lowest order terms in ϵ'.

First writing $H = \exp\{-(\epsilon'/V)h(z)\}$, where $h(z) = z+\sqrt{(z^2+1)}-1$, >0 for all $z > 0$,

$$J_1(\epsilon') = \epsilon' \int_{V=0}^{\delta(\epsilon')}\int_{z=0}^{\infty} V^2 e^{-V^2}(G-1)\exp\left\{-\left(\frac{\epsilon'}{V}\right)h(z)\right\}dz\,dV$$
$$+\epsilon' \int_{V=\delta}^{\infty}\int_{z=0}^{\infty} V^2 e^{-V^2}(G-1)\exp\left\{-\left(\frac{\epsilon'}{V}\right)h(z)\right\}dz\,dV.$$

Replacing $(G-1)$ by an upper bound and using (A-1), we see that the first term on the right is of order $\epsilon'(\delta^4/\epsilon')$. In the second term the coefficient in front of $h(z)$ is $\epsilon'/V \leqslant \epsilon'/\delta$. So if we can choose $\delta(\epsilon')$ such that $\delta^4/\epsilon' \to 0$ and $\epsilon'/\delta \to 0$ as $\epsilon' \to 0$, which is true whenever $\delta = \epsilon'^m$, $\frac{1}{4} < m < 1$, we can write

$$J_1(\epsilon') = \epsilon' \int_0^{\infty}\int_0^{\infty} V^2 e^{-V^2}(G-1)\,dz\,dV + o(\epsilon').$$

Similarly

$$J_2(\epsilon') = \epsilon'\left[\int_{V=0}^{\delta}\int_0^{\infty} V^2 e^{-V^2}\exp\left\{-\left(\frac{\epsilon'}{V}\right)h(z)\right\}dz\,dV\right.$$
$$\left.+\int_{\delta}^{\infty}\int_0^{\infty} V^2 e^{-V^2}e^{\epsilon'/V}\exp\left\{-\left(\frac{\epsilon'}{V}\right)[z+\sqrt{(z^2+1)}]\right\}dz\,dV\right]$$
$$= \epsilon' O\left(\frac{\delta^4}{\epsilon'}\right) + \epsilon' \int_{\delta}^{\infty} V^2 e^{-V^2}\left(1+\frac{\epsilon'}{V}+\dots\right)$$
$$\times \int_0^{\infty}\exp\left\{-\left(\frac{\epsilon'}{V}\right)[z+\sqrt{(z^2+1)}]\right\}\left(\frac{dz}{V}\right)V\,dV.$$

Using the result (A-1) again

$$J_2(\epsilon') = O(\delta^4) + \int_{\delta}^{\infty} V^3 e^{-V^2}\left(1+\frac{\epsilon'}{V}+\dots\right)\left\{\tfrac{1}{2}+O\left(\frac{\epsilon'^2}{V^2}\ln\frac{\epsilon'}{V}\right)\right\}dV.$$

It is clear that we can choose δ such that both δ^4 and ϵ'^2/δ^2 are $o(\epsilon')$ by taking $\delta = \epsilon'^m$, $\frac{1}{4} < m < \frac{1}{2}$. Thus

$$J_2(\epsilon') = \tfrac{1}{4}+\tfrac{1}{8}\sqrt{\pi}\epsilon' + o(\epsilon').$$

Putting these results in (A-3)

$$\dot{m} = 2\rho_1\bar{c}_1\left[\int_0^{\frac{1}{2}\pi}\left\{\tfrac{1}{4}+\tfrac{1}{8}\sqrt{\pi}\epsilon' + \epsilon'\int_0^{\infty}\int_0^{\infty} V^2 e^{-V^2}(G-1)\,dz\,dV\right\}\cos\theta\sin\theta\,d\theta\right]$$
$$= \tfrac{1}{4}\rho_1\bar{c}_1\left[1+\epsilon\left\{\tfrac{1}{2}+\tfrac{1}{8}\sqrt{\pi}\int_0^{\infty}\int_0^{\infty}\int_0^{\frac{1}{2}\pi} V^2 e^{-V^2}(G-1)\cos\theta\sin\theta\,d\theta\,dz\,dV\right\}\right], \quad \text{(A-5)}$$

which, after expanding $g_2 = \exp(h_1 V + h_2 V^2)$ as before, leads to exactly the same result as (3.20), thus justifying the splitting of the integral used in (3.11) and (3.11*a*).

A similar analysis will show that (3.11) and (3.11*a*) give correct results for all higher moments also.

REFERENCES

Bhatnagar, P. L., Gross, E. P. & Krook, M. 1954 *Phys. Rev.* **94**, 511.

Bromwich, T. J. I'A 1926 *Introduction to the Theory of Infinite Series*. London: Macmillan.

Chapman, Sydney, & Cowling, T. G. 1952 *The Mathematical Theory of Non-Uniform Gases*. Cambridge University Press.

Heuman, C. 1941 *J. Math. Phys.* **20**, 127.

Krook, M. 1959 *J. Fluid Mech.* **6**, 523.

Liepmann, H. W. 1960 *J. Fluid Mech.* **10**, 65.

Narasimha, R. 1960 *Phys. Fluids*, **3**, 476.

Present, R. D. 1958 *Kinetic Theory of Gases*. New York: McGraw-Hill.

Sadowsky, M. A. & Sternberg, E. 1950 *Quart. Appl. Math.* **8**, 113.

Willis, D. R. 1958 *Princeton Univ. Rep.* no. 442.